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A constant mean curvature surface and the Dirac operator

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Abstract. The shape of a surface with constant mean curvature (CMC) has been studied in mathematics and physics related to nonlinear integrable theory and harmonic map (σ-model) theory. In the study a fictitious (linear) Dirac-type operator appears as a tool of the calculus (Konopelchenko B G and Taimanov I A 1996 *J. Phys. A: Math. Gen.* **29** 1261–5).

In this paper, I confine the Dirac field defined in $\mathbb{R}^3$ to a thin surface embedded in $\mathbb{R}^3$ and obtain a proper Dirac operator for the thin surface. Then it completely agrees with the Dirac-type operator used in the calculus of the CMC surface theory. In other words, the mathematical Dirac-type operator is realized by a physical Dirac particle.

1. Introduction

Symmetry (transformation group) is the most important subject in physics and mathematics, as Klein declared in 1872 at Erlangen [1]. From category theory, it is well known that there is a functor (correspondence) between the analytical and differential geometrical categories if they come from the same symmetry [2]. For example, in the Atiyah–Singer index theorem [3–5] the structure of the Dirac field defined on a fibre bundle and that of the connection of the fibre bundle, i.e. the gauge field, exhibit the same symmetry and then their homomorphism to an integer are identified; the topological excitation of the gauge field and that of the Dirac field have a certain correspondence [6], as Heisenberg's matrix dynamics and Schrödinger's differential equation represent the same quantum mechanics. Due to the existence of such a functor, a bosonization (boson-fermion correspondence) scheme sometimes appears in physics.

Recently, a constant mean curvature surface, realized physically in the case of an equilibrium soap film dividing spaces with different pressure, has been studied [7–12]; the mean curvature consists of an extrinsic connection form (Weingarten map) [13, 14] and is a feature of a submanifold or immersed object. In the calculus of such a surface, a Dirac-type operator appears, and its properties exhibit the symmetry of the system [7, 8], though its prototype was founded by Weierstrass and Enneper before the discovery of the Dirac operator [7, 8, 13, 14]. The question arises as to why the Dirac operator appears in the differential geometrical problem. The purpose of this paper is to answer the question in terms of submanifold quantum mechanics.

The submanifold quantum mechanics I call upon was first discovered by Jensen and Koppe in 1971 [15]. It was rediscovered by da Costa in 1982 [16]. They considered a quantum particle confined in a low-dimensional subspace in $\mathbb{R}^3$ and then found the canonical Laplacian by constructing the Schrödinger equation in the subspace.

Following their formulation, I have been studying the canonical differential operators defined properly in a 'submanifold' embedded or immersed in $\mathbb{R}^n$, e.g., the Schrödinger, Klein–Gordon and Dirac operators [17–24]. They have potential terms in each equation consisting of the extrinsic curvature of the system.

In a series of works [17–24], I showed that the Dirac operator in a space curve immersed in $\mathbb{R}^n$ can be regarded as the Lax operator of the generalized modified Korteweg–de Vries (MKdV) equation [20], while the extrinsic curvature of a space curve obeys such a soliton equation. In other words, I found a physical (geometrical) realization of the correspondence between the infinite linear equation as a fermionic (Grassmannian) space and the nonlinear differential equation [23]. This correspondence is expected to be interpreted as a functor between the fermionic (analytic) and bosonic (differential geometrical) categories. As there had been no theory of the index theorem on the immersed space prior to [19, 22], I thus proved that the Dirac operator in submanifold quantum mechanics is a canonical object and its analytic index agrees with the topological index of the immersed space [19, 22]. In other words, I discovered the index theorem related to the immersed object [19, 22].

Furthermore, following my formulation [17], Burgess and Jensen generalized the Dirac operator in a space curve in $\mathbb{R}^2$ to that on a surface in $\mathbb{R}^3$ [25]. As do the Schrödinger operator and Klein–Gordon operator, it also includes a potential term consisting of the mean curvature.

After Polyakov pointed out that the extrinsic curvature is necessary for renormalizability in two-dimensional gravity [26], a surface with an extrinsic curvature has also been studied in elementary particle physics [27]. However, in classical theory the extrinsic curvature has played central roles because a classical object has a finite thickness [28]. In the eighteenth century, the shape of an elastica, an elastic rod with an infinitesimal thickness, was of the greatest concern in mathematical physics [28]. Due to the thickness, its Lagrangian is proportional to the square of the extrinsic curvature. Daniel Bernoulli investigated it and discovered the Lagrangian and the action principle and Euler completely classified the shape of a static elastica in $\mathbb{R}^2$. However, the dynamics of the elastica is still an open problem and its extrinsic curvature gives fruitful information about nonlinear differential equations [29–31]. Next, the shape of an equilibrium soap film under equal pressure was one of the most important problems in nineteenth century, this being known as the minimal surface problem or the Plateau problem [13, 14, 32]. The mean curvature of such a surface vanishes due to the surface tension from the Laplace formula and the surface tension comes from the existence of the thickness of the surface [32]. The constant mean curvature surface is regarded as a natural generalization of the minimal surface [7–12]. Accordingly, it is natural that in order to investigate the physical effect of the extrinsic curvature I assume an object has a thickness.

In section 2 I will review the geometrical situation of the system. In section 3 I will confine the Dirac field defined in $\mathbb{R}^3$ to an immersed thin surface following [25], and properly obtain the Dirac operator there. After the surface is restricted to a conformal flat one, in section 4 I will show that the Dirac operator defined in the immersed surface completely agrees with the Dirac-type operator in [7, 8], which was introduced as a tool to investigate the constant mean curvature surface. In other words, in this paper I will investigate the effect of the thickness of the surface by dealing with the Dirac field confined there, while the studies in differential geometry are performed by dealing with its geometrical structure [7–12]. Then I will conclude that both are compatible. At the end of section 4 I will discuss my results and give open issues related to this model.

2. The geometry of a curved surface embedded in $\mathbb{R}^3$

Before I define the problem of a Dirac field constrained to lie on a curved surface embedded in $\mathbb{R}^3$, I will set up the geometrical situation of the system. In this section, I will review the geometry of a surface $\mathcal{S}$ embedded in $\mathbb{R}^3$ [12, 13, 18]. Since the imaginary time and Euclidean quantum mechanics are very useful in the path integral method, I will deal with only the Euclidean Dirac field in this paper. Furthermore, for the sake of simplicity, I assume that such a surface is embedded in $\mathbb{R}^3$ rather than immersed temporarily.

As I will use a confinement potential m_{conf} to constrain the particle to be on $\mathcal{S}$, it also enables us to consider only the geometry in the vicinity of $\mathcal{S}$ or a tubular neighbourhood $\mathcal{T}$ of the surface $\mathcal{S}$ even before I take a limit. Due to the properties of the tubular neighbourhood $\mathcal{T}$ of the surface $\mathcal{S}$ and the affine structure of $\mathbb{R}^3$, the geometry is reduced to a simple one.

Since I wish to express a metric $G_{\mu\nu}$ in the tubular neighbourhood $\mathcal{T}$ in terms of the variables of $\mathcal{S}$, I will define the general coordinates (q^1, q^2, q^3), in terms of which the curved surface $\mathcal{S}$ will be expressed after one of its degrees of freedom has been suppressed. Let the middle parts of the Greek indices $(q^\mu, q^\nu, \ldots)$ indicate the three-dimensional curved system, $\mu = 1, 2, 3$, and the first and second coordinates indicate the position attached to $\mathcal{S}$. The relation between the Cartesian (X^1, X^2, X^3) and general coordinate systems is given through the moving frame (Jacobian element):

$$E^I{}_\mu := \partial_\mu X^I \tag{2.1}$$

where $\partial_\mu := \partial/\partial q^\mu$. The inverse matrix of $E^I{}_\mu$ is denoted by $E^\mu{}_I$. The metric is written as

$$G_{\mu\nu} := \delta_{IJ} E^I{}_\mu E^J{}_\nu. \tag{2.2}$$

When a position on $\mathcal{S}$ is represented using the affine vector $\boldsymbol{x}(q^1, q^2)$ in $\mathbb{R}^3$ and the normal unit vector of S is denoted by $\boldsymbol{e}_3$, I can uniquely express a point $\boldsymbol{X} := (X^1, X^2, X^3)$ in the vicinity of $\mathcal{S}$ in terms of them:

$$\boldsymbol{X}(q^\mu) = \boldsymbol{x}(q^\alpha) + q^3 \boldsymbol{e}_3. \tag{2.3}$$

(I will deal with only a tubular neighbourhood $\mathcal{T}$ in which I can uniquely determine a coordinate system such as (2.3).) The beginning parts of Greek indices $(q^\alpha, q^\beta, \ldots)$ span from one to two. I define the moving frame along $\mathcal{S}$ as

$$e^I{}_\alpha := \partial_\alpha x^I \tag{2.4}$$

and its inverse matrix as $e^\alpha{}_I$. I divide the ordinary derivative along $\mathcal{S}$ into the horizontal and vertical part; the horizontal part is written by ∇_α defined as

$$\nabla_\alpha \boldsymbol{b} := \partial_\alpha \boldsymbol{b} - \langle \partial_\alpha \boldsymbol{b}, \boldsymbol{e}_3 \rangle \boldsymbol{e}_3 \tag{2.5}$$

for a vector field $\boldsymbol{b}$. Here $\langle\ ,\ \rangle$ denotes the canonical inner product in the Euclidean space $\mathbb{R}^3$. The two-dimensional Christoffel symbol $\gamma^\gamma{}_{\beta\alpha}$ attached to $\mathcal{S}$ is thus defined as

$$\nabla_\alpha \boldsymbol{e}_\beta = \gamma^\gamma{}_{\beta\alpha} \boldsymbol{e}_\gamma. \tag{2.6}$$

The second fundamental form is denoted as

$$\gamma^3{}_{\beta\alpha} := \langle \boldsymbol{e}_3, \partial_\alpha \boldsymbol{e}_\beta \rangle. \tag{2.7}$$

On the other hand, from the relation $\langle \boldsymbol{e}_3, \partial_\alpha \boldsymbol{e}_3 \rangle = 0$, the Weingarten map, $-\gamma^\alpha{}_{\beta 3} \boldsymbol{e}_\alpha$, is defined by

$$\gamma^\alpha{}_{\beta 3} \boldsymbol{e}_\alpha := \nabla_\beta \boldsymbol{e}_3 \qquad \gamma^\alpha{}_{\beta 3} = \langle \boldsymbol{e}^\alpha, \partial_\beta \boldsymbol{e}_3 \rangle. \tag{2.8}$$

Because of $\partial_\alpha \langle e^\gamma, e_3\rangle = 0$, ${\gamma^\alpha}_{\beta 3}$ is associated with the second fundamental form through the relation

$$ {\gamma^3}_{\beta\alpha} = -{\gamma^\gamma}_{3\alpha} g_{\gamma\beta} \tag{2.9} $$

where $g_{\alpha\beta} := \delta_{IJ} {e^I}_\alpha {e^J}_\beta$ is the surface metric. It should be noted that for the dilation of $(q^1, q^2) \to \lambda(q^1, q^2)$, the Weingarten map does not change.

I can therefore express ${E^I}_\mu$ $(= \partial x^I/\partial q^\mu)$ in the tubular neighbourhood $\mathcal{T}$ in terms of ${e^I}_\alpha$:

$$ {E^I}_\alpha = {e^I}_\alpha + q^3 {\gamma^\beta}_{3\alpha} {e^I}_\beta. \tag{2.10} $$

The metric in the tubular neighbourhood $\mathcal{T}$ (2.2) is explicitly expressed as

$$ \begin{aligned} G_{\alpha\beta} &= g_{\alpha\beta} + [{\gamma^\gamma}_{3\alpha} g_{\gamma\beta} + g_{\alpha\gamma}{\gamma^\gamma}_{3\beta}] q^3 + [{\gamma^\gamma}_{3\alpha} g_{\gamma\delta} {\gamma^\delta}_{3\beta}](q^3)^2 \\ G_{3\alpha} &= G_{\alpha 3} = 0 \\ G_{33} &= 1 \end{aligned} \tag{2.11} $$

and $G := \det(G_{\mu\nu})$ becomes

$$ G = g\zeta \qquad \zeta^{1/2} := (1 + \mathrm{tr}_2\,({\gamma^\alpha}_{3\beta}) q^3 + \det{}_2\,({\gamma^\alpha}_{3\beta})(q^3)^2) \tag{2.12} $$

where $g := \det_2 (g_{\mu\nu})$. Here tr_2 and $\det_2$ are the two-dimensional trace and determinant over α and β, respectively. These values are invariant for the coordinate transformation if I fix the surface $\mathcal{S}$ and they are known as the mean and the Gaussian curvatures on $\mathcal{S}$:

$$ H := -\tfrac{1}{2}\,\mathrm{tr}_2\,({\gamma^\alpha}_{3\beta}) \qquad K := \det{}_2\,({\gamma^\alpha}_{3\beta}). \tag{2.13} $$

Accordingly the Christoffel symbols associated with the coordinate system of the tubular neighbourhood $\mathcal{T}$ are given as

$$ {\Gamma^\mu}_{\nu\rho} := \tfrac{1}{2} G^{\mu\tau}(G_{\nu\tau,\rho} + G_{\rho,\tau,\nu} - G_{\nu\rho,\tau}) \tag{2.14} $$

and in terms of them the covariant derivative $\overline{\nabla}_\mu$ in $\mathcal{T}$ is naturally defined as

$$ \overline{\nabla}_\mu B_\nu := \partial_\mu B_\nu - {\Gamma^\lambda}_{\mu\nu} B_\lambda \tag{2.15} $$

for a covariant vector B_μ.

3. The Dirac operator on a curved surface embedded in $\mathbb{R}^3$

As I have finished the geometrical preliminaries, in this section I will consider the Dirac field $\boldsymbol{\Psi} =: (\boldsymbol{\Psi}^1, \boldsymbol{\Psi}^2)^T$ defined in the tubular neighbourhood $\mathcal{T}$ and confine it to $\mathcal{S}$ by taking a limit [25].

I will start with the original Lagrangian given by

$$ \mathcal{L}\mathrm{d}^3x = \overline{\boldsymbol{\Psi}}(x)\mathrm{i}\left(\Gamma^I\partial_I - m_{\mathrm{conf}}(q^3)\right)\boldsymbol{\Psi}(x)\,\mathrm{d}^3x \tag{3.1} $$

where Γ^I is the gamma matrix in the three-dimensional Euclidean space $\mathbb{R}^3$, $\partial_I := \partial/\partial x^I$ and $\overline{\boldsymbol{\Psi}} = \boldsymbol{\Psi}^\dagger\Gamma^1$. Let us assume that m_{conf} has the form

$$ m_{\mathrm{conf}}(q^3) := \sqrt{\mu_0^2 + \omega^2(q^3)^2} $$

for a large ω and μ_0 with $\mu_0 \gg \sqrt{\omega}$ [24]. Since the confinement potential depends only upon the normal direction with an induced metric from that of $\mathbb{R}^3$, I can recognize that the thin surface has the same thickness all over the surface $\mathcal{S}$. Due to the confinement potential, I need not pay any attention at all to singularities of the curved coordinate system

q^μ like the origin of the spherical coordinate. Precisely speaking, since $1/\sqrt{\omega}$ is a unit of the thickness of the potential, we assume the situation, $\mu_0 \gg \sqrt{\omega}$ and $\sqrt{\omega} \gg \max_{\mathcal{S},\alpha,\beta} \gamma^{\alpha}_{3\beta}$. I note that this potential $m_{\rm conf}$ is not coupled with Γ^0 and I can avoid the disease of the Klein paradox on the confinement [17, 19]. Furthermore, since the mass potential $m_{\rm conf}$ is an even function of q^3, there is no non-trivial zero mode along the normal direction and in the process of the confinement the chirality of the Dirac field is preserved [24, 33, 34]. Thus the normal component of the Dirac field is factorized. Each energy level of the normal component splits and can be regarded as effective mass indexed by a non-negative integer. The Lagrangian can then be expressed by the sum of ones corresponding to each level. The ground state of the normal direction exists as a mode of the lowest level and is localized in the region $(-1/\sqrt{\omega}, 1/\sqrt{\omega})$ for the normal direction. By paying attention only to the ground state, the Dirac particle is approximately confined to the thin tubular neighbourhood $\mathcal{T}_0 \approx (-1/\sqrt{\omega}, 1/\sqrt{\omega}) \times \mathcal{S}$. After taking the squeezed limit, I can realize the quasi-two-dimensional subspace in $\mathbb{R}^3$ and, by integrating the Dirac field along the normal direction, express the system using the two-dimensional parameters (q^1, q^2). Then I will interpret the subspace as the surface $\mathcal{S}$ itself [17–25].

Thus I express the Lagrangian in terms of the curved coordinate system introduced in the previous section. For the coordinate transformation (2.1), the Dirac operator [19, 24, 35] becomes

$$\mathrm{i}\Gamma^I \partial_I = \mathrm{i}\Gamma^\mu \partial_\mu \tag{3.2}$$

and the spinor representation of the coordinate transformation is given as

$$\mathbf{\Psi}(q) = \mathrm{e}^{-\Sigma^{IJ}\Omega_{IJ}} \mathbf{\Psi}(x) \tag{3.3}$$

where Σ^{IJ} is the spin matrix

$$\Sigma^{IJ} := \tfrac{1}{2}[\Gamma^I, \Gamma^J] \tag{3.4}$$

and $\Sigma^{IJ}\Omega_{IJ}$ is a solution of the differential equation

$$\partial_\mu \left(\Sigma^{IJ}\Omega_{IJ}\right) = \Omega_\mu \qquad \Omega_\mu := \tfrac{1}{2}\Sigma^{IJ} E_I{}^\nu (\overline{\nabla}_\mu E_{J\nu}). \tag{3.5}$$

Hence the Lagrangian density (3.1) can be expressed in terms of the general coordinate

$$\mathcal{L}\,\mathrm{d}^3x = \overline{\mathbf{\Psi}}(q)\mathrm{i}\left(\Gamma^\mu D_\mu - m_{\rm conf}(q^3)\right)\mathbf{\Psi}(q)\,\sqrt{G}\,\mathrm{d}^3q \tag{3.6}$$

where $\Gamma^\mu := \Gamma^I E_I{}^\mu$ and D_μ denotes the spin connection

$$D_\mu := (\partial_\mu + \Omega_\mu). \tag{3.7}$$

After straightforward calculation, the spin connections [25] become

$$D_\alpha = \partial_\alpha + \Omega_\alpha \qquad D_3 = \partial_3. \tag{3.8}$$

I will follow the argument of da Costa [16–24, 36]. Since the measure on the curved system is given as

$$\mathrm{d}^3x = \sqrt{G}\cdot\mathrm{d}^3q \tag{3.9}$$

and $-\mathrm{i}\partial_3$ is neither Hermitian nor a momentum operator [36], I redefine the field as

$$\Psi = \zeta^{1/2}\mathbf{\Psi}. \tag{3.10}$$

Then the Lagrangian density (3.6) changes as

$$\mathcal{L}\,\mathrm{d}^3x = \overline{\Psi}(q)\mathrm{i}(\Gamma^\mu \mathbb{D}_\mu - m_{\rm conf}(q^3))\Psi(q)\,\sqrt{g}\,\mathrm{d}^3q \tag{3.11}$$

where

$$\mathbb{D}_\alpha := D_\alpha - \frac{1}{4}\partial_\alpha \log\zeta \qquad \mathbb{D}_3 := \partial_3 + \frac{H - Gq^3}{1 - 2Hq^3 + G(q^3)^2}. \tag{3.12}$$

In the deformed Hilbert space spanned by Ψ, $-\mathrm{i}\partial_3$ is the momentum operator and represents the translation along the normal direction.

Due to the confinement potential $m_{\rm conf}$, the Dirac field for the normal direction is factorized and can be expressed by the modes classified by a non-negative integer n [19, 24]. Then there exists a mode ($n = 0$) with the lowest energy for the normal direction such that

$$\Psi(q^1, q^2, q^3) \sim \sqrt{\delta(q^3)}\psi(q^1, q^2) \tag{3.13}$$

and

$$(\Gamma^3\partial_3 - m_{\rm conf}(q^3))\sqrt{\delta(q^3)}\psi(q^1, q^2) = m_0\sqrt{\delta(q^3)}\psi(q^1, q^2). \tag{3.14}$$

The Lagrangian density is decomposed to those classified by the mode n [19, 24]. Hence it can be regarded that each mode is an independent field. By restricting the function space of the Dirac field to that with $n = 0$ mode, the Lagrangian density on a surface $\mathcal{S}$ is defined as

$$\mathcal{L}_S^{(0)}\sqrt{g}\,\mathrm{d}^2q := \left(\int \mathrm{d}q^3\,\mathcal{L}\,\sqrt{G}\right)\Big|_{n=0}\,\mathrm{d}^2q \tag{3.15}$$

and then it has the form [25]

$$\mathcal{L}_S^{(0)}\sqrt{g}\,\mathrm{d}^2q = \mathrm{i}\overline{\psi}(\gamma^1\mathcal{D}_1 + \gamma^2\mathcal{D}_2 + H\gamma^3 + m_0)\psi\,\sqrt{g}\,\mathrm{d}^2q \tag{3.16}$$

where I rewrite the quantities as

$$\begin{aligned} &e^I{}_\alpha \equiv E^I{}_\alpha|_{q^3=0} \qquad \gamma^\mu(q^1, q^2) := \Gamma^\mu(q^1, q^2, q^3 \equiv 0) \\ &\omega_\alpha(q^1, q^2) := \Omega_\alpha(q^1, q^2, q^3 \equiv 0) \qquad \mathcal{D}_\alpha := \partial_\alpha + \omega_\alpha. \end{aligned} \tag{3.17}$$

Since a confined space is expressed by the two-dimensional parameter (q^1, q^2) and can be regarded as a two-dimensional space, I can identify the confined space with the surface $\mathcal{S}$ itself and then equation (3.16) is interpreted as a Dirac operator in a surface $\mathcal{S}$ embedded in $\mathbb{R}^3$. Hence the inner space can be written as

$$\omega_\alpha := \tfrac{1}{2}\Sigma^{ij}e_i{}^\beta(\nabla_\alpha e_{j\beta}) \tag{3.18}$$

where the indices of i, j run from 1 to 2.

It should be noted that this Dirac operator is not Hermitian in general, since neither is that in a space curve immersed in $\mathbb{R}^3$ [15–26]. It is natural because the extra term appearing as an immersed effect in the Schrödinger equation behaves as the negative potential if one confines a Schrödinger particle in a lower dimensional object [15–25]; roughly speaking the square root of the negative potential appears as a pure imaginary extra field in the Dirac equation [17–25].

Next I will mention the immersion of the surface. If there is a crossing in a surface, the crossing can be moved in another direction by embedding $\mathcal{S}$ in a higher-dimensional space, e.g., $\mathbb{R}^4$, as long as it does have a negligible effect on the curvature. Imagine that a thin rope is on a table (two-dimensional space) with a crossing. To be exact such a crossing is not a mathematical crossing because of the thickness of the rope in three-dimensional space, even though its shape can be parameterized by two-dimensional space. Similarly, I can extend the set of the surface $\{\mathcal{S}\}$, which has been embedded in $\mathbb{R}^3$, to ones roughly immersed in $\mathbb{R}^3$ and precisely embedded in $\mathbb{R}^4$ without mathematical crossing. Then such an operation will not affect the above computations at all if the thickness of the surface is appropriate. Thus I can extend the Lagrangian density (3.16) to that of a surface immersed in $\mathbb{R}^3$.

4. The Dirac operator on a complex surface immersed in $\mathbb{R}^3$

In general m_0 does not vanish, but since I am interested in the properties of the Dirac operator itself, I will neglect the mass m_0 hereafter; I will investigate the high energy behaviour of the field with m_0, i.e. the behaviour of the high energy of the surface direction and the lowest energy of the normal direction using the independence of the normal modes.

In this section, I will consider only a surface with the conformal flat metric [26, pages 228–35] immersed in $\mathbb{R}^3$:

$$g_{\alpha\beta}\mathrm{d}q^\alpha \mathrm{d}q^\beta = \rho\delta_{\alpha,\beta}\mathrm{d}q^\alpha\mathrm{d}q^\beta. \tag{4.1}$$

This condition looks very strong but is very natural. Surfaces appearing in physics are sometimes complex analytic and are conformal flat. Then the Christoffel symbol is calculated as

$$\gamma^\alpha_{\ \beta\gamma} = \tfrac{1}{2}\rho^{-1}(\delta^\alpha_{\ \beta}\partial_\gamma\rho + \delta^\alpha_{\ \gamma}\partial_\beta\rho - \delta_{\beta\gamma}\partial_\alpha\rho). \tag{4.2}$$

We have a natural Euclidean inner space denoted by the parameters y^a, y^b, …, $(a, b = 1, 2)$. Then the moving frame is written as

$$e^a_{\ \alpha} = \rho^{1/2}\delta^a_{\ \alpha} \tag{4.3}$$

and the gamma matrix is connected to the flat one σ^a:

$$\gamma^\alpha = e_a^{\ \alpha}\sigma^a. \tag{4.4}$$

Thus the spin connection becomes

$$\omega_\alpha = -\tfrac{1}{4}\rho^{-1}\sigma^{ab}(\partial_a\rho\delta_{\alpha b} - \partial_b\rho\delta_{\alpha a}) \tag{4.5}$$

where $\sigma^{ab} := \frac{1}{2}[\sigma^a, \sigma^b]$. The Dirac operator can be expressed as

$$\gamma^\alpha\mathcal{D}_\alpha = \sigma^a\delta^\alpha_{\ a}\left[\rho^{-1/2}\partial_\alpha + \tfrac{1}{2}\rho^{-3/2}(\partial_\alpha\rho)\right]. \tag{4.6}$$

Similarly to (3.10), I will redefine the Dirac field in the surface $\mathcal{S}$ as

$$\psi := \rho^{1/2}\boldsymbol{\psi} \tag{4.7}$$

and then the Dirac operator (4.6) becomes simpler [26]:

$$\gamma^\alpha\mathcal{D}_\alpha\boldsymbol{\psi} = \rho^{-1}\sigma^a\delta^\alpha_{\ a}\partial_\alpha\psi. \tag{4.8}$$

Noting that the metric of the direction q^3 is unit, $G_{33} = 1$ and now q^3-direction is inner space, $\gamma^{\alpha=3}$ should be regarded as

$$\gamma^{\alpha=3} \equiv \sigma^{a=3} \tag{4.9}$$

where $\sigma^3 := -\mathrm{i}\sigma^1\sigma^2$. Since $\overline{\boldsymbol{\psi}}\gamma^1\boldsymbol{\psi}\sqrt{g}\ \mathrm{d}^2q$ is the charge density, I expect the relation

$$\overline{\psi} = \psi^\dagger\gamma_1 = \psi^\dagger\sigma^1\rho^{1/2}. \tag{4.10}$$

Then the Lagrangian density (3.16) is reduced to

$$\mathcal{L}_S^{(0)}\sqrt{g}\ \mathrm{d}^2q = \mathrm{i}\overline{\psi}\rho^{-1/2}(\sigma^a\delta^\alpha_{\ a}\partial_\alpha + \rho^{1/2}H\sigma^3)\psi\ \mathrm{d}^2q. \tag{4.11}$$

If the complex parameterization of the surface [7, 8]

$$z := q^1 + \mathrm{i}q^2 \tag{4.12}$$

is employed, and

$$\partial := \tfrac{1}{2}(\partial_{q^1} - \mathrm{i}\partial_{q^2}) \qquad \bar\partial := \tfrac{1}{2}(\partial_{q^1} + \mathrm{i}\partial_{q^2}) \tag{4.13}$$

the Lagrangian density (4.11) is explicitly expressed as

$$\mathcal{L}_S^{(0)}\sqrt{g}\,\mathrm{d}^2q = \left(\psi_+^*\partial\psi_+ + \psi_-^*\bar\partial\psi_- - \tfrac{1}{2}\rho^{1/2}H(\psi_+^*\psi_- - \psi_-^*\psi_+)\right)\mathrm{d}^2z \qquad (4.14)$$

where $\psi_\pm^*$ is not the complex conjugate of $\psi_\pm$ and $2\mathrm{i}\,\mathrm{d}^2q \equiv \mathrm{d}\bar{z}\wedge\mathrm{d}z =: \mathrm{d}^2z$.

Hence the equation of motion of the Dirac field in the complex surface $\mathcal{S}$ immersed in $\mathbb{R}^3$ is derived as

$$\partial\psi_+ = p(z,\bar{z})\psi_- \qquad \bar\partial\psi_- = -p(z,\bar{z})\psi_+ \qquad (4.15)$$

$$\partial\psi_+^* = p(z,\bar{z})\psi_-^* \qquad \bar\partial\psi_-^* = -p(z,\bar{z})\psi_+^* \qquad (4.16)$$

where the 'external' field p is defined as

$$p := \tfrac{1}{2}\rho^{1/2}H. \qquad (4.17)$$

From (4.15) and (4.16), under the on-shell condition, the identity

$$\begin{pmatrix}\psi_+^*\\ \psi_-^*\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\overline{\psi}_+\\ \overline{\psi}_-\end{pmatrix} \qquad (4.18)$$

holds, where $\overline{\psi}_\pm$ is the complex conjugate of $\psi_\pm$. Then it is very surprising that equations (4.15) are completely identified with equation (1) of Konopelchenko and Taimanov [7, 8], introduced as tools for a calculus of the immersed complex surface in $\mathbb{R}^3$. In other words, the Dirac operator introduced by Konopelchenko in [7] can be interpreted as that in the surface given through the procedure of submanifold quantum mechanics.

In a special situation, I can allow that the y's and x's are identified as

$$x^1 \equiv y^1 \qquad x^2 \equiv y^2 \qquad (4.19)$$

but in the general case their relation cannot be easily described. However, the extrinsic geometry can be expressed by the Dirac field itself. When I use the complex parameterization of the part of the Euclidean space

$$Z := x^1 + \mathrm{i}x^2 \qquad (4.20)$$

we have the special solutions of (4.15)

$$2\mathrm{i}(\psi_+)^2 := -\bar\partial\,\overline{Z} \qquad 2\mathrm{i}(\psi_-)^2 := \partial\overline{Z} \qquad -2\psi_+\psi_- = \partial x^3 \qquad (4.21)$$

which were derived in [7]. Precisely speaking, equations (4.21) are compatible with (4.15)–(4.18), but should be interpreted as formulae which exhibit the connection between the Dirac fields and the geometry. Thus these may be regarded as a new bosonization of the fermionic field [23, 26, 29, 30], since I obtained a similar formulation for the elastica problem by complexifying its arclength [29]. In the elastica problem, the square root of the normal and tangential vector can be regarded as the Dirac field with a half-spin and is related to the vertex operator in soliton physics [29]. As Konopelchenko mentioned [7], the correspondence (4.21) comes from the Frenet–Serret relation or Weierstrass–Enneper relation. As modern representations of the Frenet–Serret relation related to soliton theory were given by Goldstein and Petrich [37], I found that they are, also, closely related to elementary particle physics and algebraic geometry [29, 30].

Thus if one solves (4.15) under a certain condition, one also finds the shape of the surface. In other words, the analytical properties of the Dirac field determine the geometry of the system.

The dilation factor is obtained by straightforward computation [7] as

$$\rho^{1/2} = 2(\psi_+^*\psi_- - \psi_-^*\psi_+) = 2(\overline{\psi}_+\psi_+ + \overline{\psi}_-\psi_-) \qquad (4.22)$$

and the Gauss curvature is calculated as

$$K = -2\frac{\partial\bar{\partial}\log(\rho)}{\rho}. \tag{4.23}$$

The Euler number can be calculated by the formula [7]

$$\chi = \frac{1}{2\pi}\int K\rho\, \mathrm{d}z\wedge \mathrm{d}\bar{z} = -\frac{1}{2\pi}\int \bar{\partial}\rho^{-1/2}\partial[2(\psi_+^*\psi_- - \psi_-^*\psi_+)]\, \mathrm{d}z\wedge \mathrm{d}\bar{z}. \tag{4.24}$$

This might be related to the index theorem [2–6, 20, 22].

For the constant mean curvature surface, (H = constant), the Dirac field is coupled with the dilaton [26], $\phi := \log\rho$:

$$\mathcal{L}_S^{(0)}\sqrt{g}\,\mathrm{d}^2q = \left(\psi_+^*\partial\psi_+ + \psi_-^*\bar{\partial}\psi_- - \tfrac{1}{2}\mathrm{e}^{\phi/2}H(\psi_+^*\psi_- - \psi_-^*\psi_+)\right)\mathrm{d}^2z. \tag{4.25}$$

This dilaton is governed by the sinh-Gordon equation [9], which is related to the Liouville equation. As Konopelchenko did, by dealing with (4.25) I can investigate the symmetrical properties of the surface. Just as with an ordinary Dirac field, it has a momentum current and a Hamiltonian. These are identified with those derived by Konopelchenko and Taimanov in [8].

When $H = 0$, the equation of motion (4.15) becomes the Weierstrass–Enneper formula [7, 8, 13, 14], which was obtained as a scheme for getting a minimal surface in the last century, and the Lagrangian (4.25) becomes that in ordinary (classical) conformal field theory. Since Weierstrass and Enneper proved that there is a one-to-one correspondence between the minimal surface and the conformal function, it can also be interpreted as a kind of fermion–boson correspondence in this theoretical framework.

As I showed the correspondence between the analytical system of the Dirac field and the geometrical system of the base space $\mathcal{S}$, I will comment on possibilities of this theory.

Since the surface is now a conformal flat one from (4.1) and the mean curvature does not directly have any effect upon the intrinsic properties of the surface, we may regard (4.14) as a natural generalization of that of conformal field theory. Even with the same mean curvature, e.g., $H = 1$, various surfaces are found [7–12]. In other words, I could classify conformal surfaces by their extrinsic properties.

Furthermore, the Dirac operator

$$\begin{pmatrix} -\rho^{1/2}H/2 & \partial \\ \bar{\partial} & \rho^{1/2}H/2 \end{pmatrix} \tag{4.26}$$

can be regarded as a generalization of that in a thin rod in $\mathbb{R}^2$ which is identified with the Lax operator of the MKdV equation [17, 19]. Since I obtained the Dirac operator associated with the generalized MKdV equation by confining a Dirac operator to a rod in $\mathbb{R}^n$ [20], I could generalize the Dirac operator (4.26) to that on a surface in $\mathbb{R}^n$ $n > 3$. Since the generalized MKdV theory is closely related to the W-algebra, the Dirac operator of a surface in $\mathbb{R}^n$ might be expressed in the framework of the Gervais and Matsuo formulation on the W-algebra [27].

In soliton theory, the momentum space of the Dirac operator can be expressed by the compact Riemannian surface with general genus. It was shown that the constant mean curvature surface can also be realized as a compact Riemannian surface [10–12]. Hence there is a kind of duality between the configuration space and the momentum space because both are described algebraically. The Dirac operator (4.26) may exhibit such duality. Thus I expect further study on the duality.

Next I will mention the dynamics of the surface $\mathcal{S}$. The Dirac operator (4.26) is applicable to a more general mean curvature H. Thus I can consider a deformation of the surface $\mathcal{S}$ preserving the whole spectrum of the Dirac operator (4.26). Such a deformation can be expressed by a partial differential equation of q^α and the deformation parameter. Then such a differential equation is expected to be regarded as a $(2+1)$-dimensional soliton equation as pointed out by Konopelchenko [38]. I believe that the dynamics of the surface is a very good toy for the physicist to use in finding the higher-dimensional soliton equation, as Euler found various solutions of the sine-Gordon equation by observing an elastica long before the discovery of the Korteweg–de Vries equation [28].

Finally, I will comment on another possibility. As I described in the introduction, thickness is the most important factor in the extrinsic curvature theory. Hence the Polyakov program for the two-dimensional gravity may be archived by assuming a thickness of the string or the two-dimensional universe. Such an assumption might open other theoretical possibilities in particle and universe physics [21]. Then studies of the deformation of the surface are again required, since if one quantizes the surface, one must then deal with the above deformation using Schwinger's proper time as a deformation parameter.

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References

[1] Klein F 1893 *Math. Ann.* **43** 63–100
[2] Grothendieck A 1957 *Tôhoku Math J.* **9** 119–221
[3] Atiyah M F and Singer I M 1968 *Ann. Math.* **87** 484–530
[4] Atiyah M F and Singer I M 1968 *Ann. Math.* **87** 546–604
[5] Gilkey P B 1984 *Invariance Theory, The Heat Equation and the Atiyah–Singer Index Theorem* (Wilmington: Publish or Perish)
[6] Alvarez-Gaumé L and Ginsparg P 1985 *Ann. Phys.* **161** 423–90
[7] Konopelchenko B G 1996 *Stud. Appl. Math.* **96** 9–51
[8] Konopelchenko B G and Taimanov I A 1996 *J. Phys. A: Math. Gen.* **29** 1261–5
[9] Pinkall U and Sterling I 1989 *Ann. Math* **130** 407–51
[10] Abresh U 1987 *J. Reine Angew. Math.* **374** 169–92
[11] Kapouleas U 1990 *Ann. Math* **131** 239–330
[12] Hitchin N J 1990 *J. Diff. Geom.* **31** 627–710
[13] Kobayashi S 1995 *Differential Geometry of Curves and Surfaces* (Tokyo: Shokabo) 2nd edn (in Japanese)
[14] Guggenheimer H W 1977 *Differential Geometry* (New York: Dover)
[15] Jensen H and Koppe H 1971 *Ann. Phys.* **63** 586–91
[16] da Costa R C T 1981 *Phys. Rev.* A **23** 1982–7
[17] Matsutani S and Tsuru H 1992 *Phys. Rev.* A **46** 1144–7
[18] Matsutani S 1993 *Phys. Rev.* A **91** 686–9
[19] Matsutani S 1994 *Prog. Theor. Phys.* **91** 1005–37
[20] Matsutani S 1994 *Phys. Lett.* **189A** 27–31
[21] Matsutani S 1994 *Nuovo Cimento* **109B** 493–7
[22] Matsutani S 1995 *J. Phys. A: Math. Gen.* **28** 1399–412
[23] Matsutani S 1995 *Int. J. Mod. Phys.* A **10** 3091–107
[24] Matsutani S 1995 *Thesis* Tokyo Metropolitan University
[25] Burgess M and Jensen B 1993 *Phys. Rev.* A **48** 1861–6
[26] Polyakov A M 1987 *Gauge Fields and Strings* (London: Harwood)
[27] Gervais J-L and Matsuo Y 1993 *Commun. Math. Phys.* **152** 317–68

[28] Love A E H 1927 *A Treatise on the Mathematical Theory of Elasticity* (Cambridge: Cambridge University Press)
[29] Matsutani S 1995 *Int. J. Mod. Phys.* A **10** 3109–30
[30] Matsutani S 1995 *Mod. Phys. Lett.* **10A** 717–21
[31] Nishinari K 1997 *Proc. R. Soc.* A to appear
[32] Landau L D and Lifshitz E M 1963 *Fluid Mechanics* (Oxford: Pergmon) 2nd edn
[33] Witten E 1982 *J. Diff. Geom.* **17** 661–92
[34] Eguchi T 1990 *Suurikagaku* **325** 45–9 (in Japanese)
[35] Birrell N D and Davies P C 1982 *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press)
[36] Suzuki A and Matsutani S 1996 *Nuovo Cimento* **111B** 593–606
[37] Goldstein R E and Petrich D M 1991 *Phys. Rev. Lett.* **67** 3203–6
[38] Konopelchenko B G 1996 Multidimensional integral systems and dynamics of surface in space *Preprint*